NOTES

A RANDOM SET PROCESS IN THE PLANE WITH A MARKOVIAN PROPERTY[1]

By Paul Switzer[2]

Harvard University

0. Introduction. In a recent paper concerning the pattern in a planar region of one species of vegetation with respect to another, Pielou (1964) defined a "random" pattern as one in which the alternation between species along any line transect is Markovian. However, Bartlett (1964) pointed out that she did not establish the existence of a two-state planar process having this Markovian property. He also noted that the correlation between a pair of points of a two-state Markov process must be $e^{-hv}(h > 0)$ where v is the distance between the points; by referring to Whittle (1954, p. 448) he claimed that in two dimensions this correlation function does not correspond to any simple random hypothesis. Whittle's point seems to be based on the somewhat complicated nature of the two-dimensional Fourier transform of an exponential function.

While there is a concensus about the non-elementary character of the exponential correlation function for processes in the plane, it has nevertheless been used to smooth empirical correlograms and as a basis to compare sampling designs; see Matérn (1960, p. 52 and p. 83). It is of course well-known that e^{-hv} does belong to the class of possible correlation functions. Especially, it is possible for the class of two-state processes since it is convex downward in a neighborhood of $v = 0$; this necessary condition was noted by Matérn (1960, p. 42) and is not satisfied, for example, by the function chosen by Whittle (1954, p. 448) as the "elementary" correlation function for general stationary and isotropic processes in the plane.

This paper demonstrates the existence of a *finite*-state random process in the plane with the property that the alternation among states along any straight line is Markovian. The process is quite elementary, and can be regarded as a simple hypothesis of randomness albeit in a somewhat special sense. Specializing to the case of *two* states, it must follow of course that the correlation function of our model has the form e^{-hv}.

1. Constructive description of the model. First, let the plane be partitioned into "cells" by locating straight lines at random according to the following procedure: Suppose the family of all straight lines in the plane is given by $x \cos \theta + y \sin \theta - p = 0$ relative to a fixed Cartesian co-ordinate system, with

Received 1 June 1965.

[1] This work was supported in part by a grant from the National Science Foundation GS-341.

[2] Now at Stanford University.

the (θ, p) parameter space being the infinite strip $\theta \varepsilon [0, \pi)$, $p \varepsilon (-\infty, \infty)$. Then lines are randomly chosen by choosing points in the parameter space according to a planar Poisson point process with average density λ per unit area. This method of choosing straight lines at random is mentioned in Kendall and Moran (1963), where some interesting properties of the method are explored.

[One can easily construct a realization of this random partition into cells for any finite subregion, R, of the plane. For example, suppose R can be enclosed by a circle of radius r; fix the origin of the plane at the center of this circle and pick an arbitrary set of x- and y-axes through the origin. Now select n points independently and at random in the rectangular set $\theta \varepsilon [0, \pi)$, $p \varepsilon (-r, r)$, where n is a Poisson random variable with mean $2\pi r\lambda$. Then the n lines corresponding to these points may intersect the finite subregion R, but it follows that lines corresponding to points outside of this rectangular parameter set will not intersect R.]

Now that one has the partition of the plane into convex polygonal cells, each of these cells is independently "colored", the selection of colors being randomly made from m available colors. The probability distribution on the colors remains the same from cell to cell, and the colors correspond of course to the states of the process.

2. The Markov property of linear transects. Let $z_1, z_2, \cdots, z_n$ be a set of ordered collinear points in the plane. Then the Markov property for linear transects will be demonstrated if we can show that

$$f(A_1 \mid A_2, \cdots, A_n) = f(A_1 \mid A_2), \tag{1}$$

where A_i, $i = 1, 2, \cdots, n$, is the color random variable at the point z_i and f is the probability function.

First it is noted that the underlying random line process induces a random partition of the sequence of n collinear points into subsequences defined as follows—two points z_i, z_{i+1} are in the same subsequence if and only if none of the random lines crosses the segment $z_i z_{i+1}$. There are 2^{n-1} possible partitions of this kind; it will be convenient to divide these into two equal classes $\{C_j\}$ and $\{C_j'\}$, $j = 1, \cdots, 2^{n-2}$, where C_j and C_j' denote partitions which differ only in that z_1 and z_2 are in different subsequences under C_j and in the same subsequence under C_j'.

For a given set of values of the state random variables $\{A_i\}$, we will say that $\{A_i\}$ is compatible with a given partition if, for any i, $A_i = A_{i+1}$ whenever z_i and z_{i+1} are in the same subsequence of the partition. Clearly, if $\{A_i\}$ is compatible with C_j' then it is compatible with C_j.

A given partition C_j may be characterized by the increasing sequence $1 = n_1, n_2, \cdots, n_r = n$ $(r \leq n)$ in that at least one random line crosses each of the segments $z_{n_i} z_{n_i+1}$, $i = 1, \cdots, r-1$, but otherwise no line crosses $z_1 z_n$. (The dependence of r and the n_i sequence on j is not indicated in this notation but should be understood.) It is clear that the corresponding partition C_j' can be characterized in a similar way by the sequence $n_2, \cdots, n_r = n$.

LEMMA 1. $\Pr(C_j)/\Pr(C_j') = e^{2\lambda v_1} - 1$, *all* j, *where* $v_i = |z_i - z_{i+1}|$, $i = 1, \cdots, n - 1$, *are the lengths of the intervals between successive points.*

According to the description of the random line process, let T be the subset in (θ, p) parameter space corresponding to the family of all lines $x \cos \theta + y \sin \theta - p = 0$ which cross a given line segment S of length v. Then it can be shown that the area of T is equal to $2v$ [see Kendall and Moran (1963, p. 58)]; hence the number of lines of the process crossing S has a Poisson distribution with mean $2v\lambda$, where λ is the density of the Poisson point process in (θ, p) space. In particular, the probability that none of the lines crosses S is just $e^{-2v\lambda}$. Furthermore, if S and S^* are non-overlapping segments of the same line then the corresponding subsets T and T^* in (θ, p) space are non-overlapping since the same random line cannot cross both segments; hence, the number of lines crossing S is independent of the number of lines crossing S^*. Using these results and the definition of C_j given above, it follows that

$$\text{(2)} \quad \Pr(C_j) = \exp\{-2\lambda(\textstyle\sum_{i=1}^{n-1} v_i - \sum_{i=1}^{r-1} v_{n_i})\} \prod_{i=1}^{r-1} [1 - \exp(-2\lambda v_{n_i})].$$

Similarly, for the corresponding partition C_j', it follows that

$$\text{(3)} \quad \Pr(C_j') = \exp\{-2\lambda(\textstyle\sum_{i=1}^{n-1} v_i - \sum_{i=2}^{r-1} v_{n_i})\} \prod_{i=2}^{r-1} [1 - \exp(-2\lambda v_{n_i})],$$

whence $\Pr(C_j)/\Pr(C_j') = \exp(2\lambda v_{n_1}) - 1$ and the lemma is established since $n_1 = 1$.

LEMMA 2. *For all* j,

$$\text{(i)} \quad f(A_1, A_2, \cdots, A_n \mid C_j') = f(A_1, A_2, \cdots, A_n \mid C_j)/f(A_1) \quad \textit{whenever} \quad A_1 = A_2$$

$$= 0 \quad \textit{whenever} \quad A_1 \neq A_2 .$$

$$\text{(ii)} \quad f(A_2, \cdots, A_n \mid C_j') = f(A_2, \cdots, A_n \mid C_j) = f(A_1, A_2, \cdots, A_n \mid C_j)/f(A_1).$$

(i) We first remark that if a set of values of $A_1, \cdots, A_n$ is not compatible with a given partition then the conditional probability of the event $(A_1, \cdots, A_n)$ is zero. This obtains because incompatibility implies that there are two points z_i, z_{i+1} falling in the same subsequence whose colors A_i, A_{i+1} are different. But it is clear that two points are in the same subsequence if and only if they are in the same basic "cell" generated by the random line process. Since the coloring process assigns the same color to the entire area of a basic cell, the event (A_i, A_{i+1}) conditional on the partition is impossible, and in particular so is $(A_1, A_2, \cdots, A_n)$. When $A_1 \neq A_2$ it is obvious that $A_1, A_2, \cdots, A_n$ is not compatible with any partition in the class $\{C_j'\}$, hence $f(A_1, A_2, \cdots, A_n \mid C_j') = 0$ for all j, as required. When $A_1 = A_2$, it follows that $A_1, A_2, \cdots, A_n$ is compatible with C_j if and only if it is compatible with C_j' for any j. If it is compatible with neither C_j and C_j' we get trivially that $f(A_1, A_2, \cdots, A_n \mid C_j') = f(A_1, A_2, \cdots, A_n \mid C_j)/f(A_1)$ since both sides are zero. If it is compatible with both C_j and C_j' the following argument applies: According to the definition of the partition C_j, the points $z_{n_i+1}, z_{n_i+2}, \cdots, z_{n_{i+1}}$ all belong to the same subsequence $(i = 0, 1, \cdots, r-1; n_0 = 0)$ hence they are all in the same basic cell of the map and must, therefore, all have the same color. So the joint event $(A_{n_i+1}, A_{n_i+2}, \cdots, A_{n_{i+1}})$ is just equivalent to the event $A_{n_{i+1}}$, say. Furthermore, the points $z_{n_1}, z_{n_2}, \cdots, z_{n_r} = z_n$ are all in different subsequences hence in different cells. Since the cells were colored independently of one another, the random variables $A_{n_1}, A_{n_2}, \cdots, A_n$ must be mutually independent. From these considerations it follows easily that

$$f(A_1, A_2, \cdots, A_n \mid C_j) = \textstyle\prod_{i=1}^{r} f(A_{n_i}), \qquad \text{and}$$

$$(4) \quad f(A_1, A_2, \cdots, A_n \mid C_j') = \textstyle\prod_{i=2}^{r} f(A_{n_i})$$
$$= f(A_1, A_2, \cdots, A_n \mid C_j)/f(A_1), \quad \text{since } n_1 = 1.$$

(ii) If $A_1, A_2, \cdots, A_n$ is not compatible with C_j then clearly $A_2, \cdots, A_n$ is not compatible with either C_j or C_j'; in this case the statement of the lemma is trivially true in that all three quantities are zero. If, however, $A_1, A_2, \cdots, A_n$ is compatible with C_j then clearly $A_2, \cdots, A_n$ is compatible with both C_j and C_j'; in this case we reason as above and find that

$$f(A_2, \cdots, A_n \mid C_j) = \textstyle\prod_{i=2}^{r} f(A_{n_i}) = f(A_2, \cdots, A_n \mid C_j')$$
$$= f(A_1, A_2, \cdots, A_n \mid C_j)/f(A_1) \qquad \text{since } n_1 = 1.$$

The Markov property (1) for linear transects can now be easily established. We have that

$$(5) \quad f(A_1, A_2, \cdots, A_n) = \textstyle\sum_j \{\Pr(C_j) f(A_1, A_2, \cdots, A_n \mid C_j) + \Pr(C_j') f(A_1, A_2, \cdots, A_n \mid C_j')\},$$

and by Lemma 1 and Lemma 2,

$$f(A_2, \cdots, A_n) = \textstyle\sum_j \{\Pr(C_j) f(A_2, \cdots, A_n \mid C_j) + \Pr(C_j') f(A_2, \cdots, A_n \mid C_j')\}$$
$$= [1 - \exp(-2\lambda v_1)]^{-1} [f(A_1)]^{-1} \textstyle\sum_j \Pr(C_j) f(A_1, A_2, \cdots, A_n \mid C_j).$$

Case 1. $A_1 \neq A_2$. In this case $f(A_1, A_2, \cdots, A_n \mid C_j') = 0$ for all j by Lemma 2, hence $f(A_1, A_2, \cdots, A_n) = \sum_j \Pr(C_j) f(A_1, A_2, \cdots, A_n \mid C_j)$; therefore,

$$(6) \qquad f(A_1 \mid A_2, \cdots, A_n) = f(A_1)[1 - \exp(-2\lambda v_1)].$$

Since this last expression is identical for all $n \geqq 2$, the Markov property (1) is established for the case $A_1 \neq A_2$.

Case 2. $A_1 = A_2$. In this case

$$f(A_1, A_2, \cdots, A_n \mid C_j') = f(A_1, A_2, \cdots, A_n \mid C_j)/f(A_1)$$

for all j by Lemma 2. Using this result and Lemma 1, we get that

$$f(A_1, A_2, \cdots, A_n) = \{1 + [f(A_1)]^{-1}[\exp(2\lambda v_1) - 1]^{-1}\} \cdot \textstyle\sum_j \Pr(C_j) f(A_1, A_2, \cdots, A_n \mid C_j),$$

therefore,

$$(7) \qquad f(A_1 \mid A_2, \cdots, A_n) = \exp(-2\lambda v_1) + f(A_1)[1 - \exp(-2\lambda v_1)].$$

This expression is also identical for all $n \geqq 2$, hence the Markov property for linear transects is established.

For the special case of two states (colors), let [0, 1] be the sample space for each of the random variables A_1 and A_2. Then the covariance function of the process is defined as $c(v_1) = E(A_1A_2) - E(A_1)E(A_2)$, where v_1 is the distance between z_1 and z_2, and the correlation function is defined as $r(v_1) = c(v_1)/c(0)$. Using Formula (7) it follows easily that $r(v_1) = \exp(-2\lambda v_1)$, as was to be expected.

REFERENCES

Bartlett, M. S. (1964). A note on spatial pattern. *Biometrics* **20** 891–892.

Kendall, M. G. and Moran, P. A. P. (1963). *Geometrical Probability*. Hafner, New York.

Matérn, B. (1960). Spatial variation. *Medd. från Statens Skogsforskningsinstitut*. **49** 1–144.

Pielou, E. C. (1964). The spatial pattern of two-phase patchworks of vegetation. *Biometrics* **20** 156–167.

Whittle, P. (1954). On stationary processes in the plane. *Biometrika* **41** 434–449.